

Regional Information Capacity of the Linear Time-Varying Channel

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Abstract—We determine the information capacity of the linear, time-varying communications channel with additive white Gaussian noise for transmission signals with support approximately restricted to closed regions of the time and frequency domains. We address the two-part problem of first, constructing appropriate transmission functions, and second, determining the mutual information. Our approach provides a signaling set that is adaptive to the time and frequency stability of the channel, and we use this set to estimate the channel's information capacity. In the limiting regime, this approach recovers the time-invariant capacity up to a redundancy factor.

I. INTRODUCTION

A. The Time-Varying Channel

We address the information capacity of the linear time-varying channel, given by the time-varying convolution

$$r(t) = \int h(t, t - \tau) s(\tau) d\tau. \quad (1)$$

We determine the maximum mutual information between the input and the output of the system, in terms of the time-varying impulse response h .

There are two ways to approach the information capacity of a channel; one may consider conditional probabilities [1], [2], or one may use the singular values of the channel's matrix representation [3], [4]. Here we take the latter approach.

With appropriate conditions on h , we view equation (1) as a bounded operator mapping $L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R})$. However, in order to use classical information theory tools, we must view the channel (1) as a map $X \rightarrow L^2(\mathbb{R})$, where X is a finite dimensional subspace of $L^2(\mathbb{R})$. If $\{e_l\}_{l \in \mathbb{Z}}$ is an orthonormal basis for $L^2(\mathbb{R})$ and $\{e_k\}_{k=1}^n$ is an orthonormal basis for X , then we represent the channel by setting

$$\mathbf{A}_{k,l} = \langle \int h(\cdot, \cdot - \tau) e_l(\tau) d\tau, e_k \rangle,$$

for $l = 1, \dots, n$, $k \in \mathbb{Z}$. Then the normalized information capacity of the channel mapping $X \rightarrow L^2(\mathbb{R})$ by (1) with additive Gaussian noise $\sim \mathcal{N}(0, \eta^2 I)$, is

$$\frac{1}{n} \sum_{k=1}^n \log(1 + \frac{\lambda_k(\mathbf{A}^* \mathbf{A})}{\eta^2}) \text{ bits}. \quad (2)$$

In the time-invariant setting, the limit of (2) is known to converge due to a theorem of Szegő [5]. For other channels, in

particular for the time-varying channel, the analogous expression to equation (2) does not *a priori* converge. Convergence questions are, therefore, one reason to first focus on the singular values of the time-varying channel when restricted to a finite dimensional subset of $L^2(\mathbb{R})$.

A second reason for considering the restriction of the channel to a finite dimensional subset is that if the function h in equation (1) has mild enough characteristics to be viewed as a function rather than a distribution, then the channel (1) is given by a compact operator. Hence, the only accumulation point of its spectrum is 0, and a limit analogous to (2) would converge to 0. This would occur even though the channel may be very robust when restricted to a finite dimensional subspace.

We consider the channel when restricted to a finite dimensional subspace and associate that space to a region of the time-frequency plane. The information capacity is then determined by the singular values of the matrix representation of the channel. We relate the singular values of this matrix to samples of a function derived from h in (1), analogously to samples of $|\hat{h}(\omega)|^2$ in the time-invariant case. In particular, we estimate the information capacity of the linear map $X \rightarrow L^2(\mathbb{R})$ for an appropriate space X in terms of a function determined by the time-varying impulse response h .

The difficulty here is determining an appropriate space X or, equivalently, an appropriate set of signaling functions. It is well known that functions with certain time-frequency localization are approximate eigenfunctions of the time-varying channel. Moreover, it is natural to ask what the achievable information capacity is for the channel (1) if the signals are restricted to a certain time-frequency region. Additionally, we impose the requirement of a structure on the set of signals.

Our work merges two aspects of research on time-varying channels. Previous authors have discussed diagonalizing the channel and giving the capacity in terms of singular values [6], [7], [8], and other authors have focused on the ideal transmission signals [9], [10], [11]. Much of the mathematical approach to time-varying channels from a time-frequency analysis perspective originated with Kozek [9], [12], [13]. While he addresses issues such as the composition and estimation of time-varying channel operators and the time-frequency localization of transmission signals, his focus is a WSUS model. Here we work with a deterministic channel and rigorously relate the channel's singular values and the signaling functions.

B. Summary of Results

Our focus is on the physical layer of the channel as given by the model (1). We show that a signaling set with appropriate time-frequency localization exists and show how it can be

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obtained. We use this signaling set to approximate the singular values of the matrix representation of the channel by its diagonal entries and then further approximate the eigenvalues by samples of a spreading function corresponding to the channel's pseudodifferential operator representation. We then approximate the information capacity of the channel for this system. These results hold for an entire class of channels given by time-frequency localization. Lastly, we show that this framework recovers the classical time-invariant information capacity up to a redundancy factor.

Note that we speak of maximum mutual information or *information capacity* rather than capacity. To prove a complete capacity result, one must prove the existence of an appropriate code. Codes, in turn, require error estimates as the code length increases. In our setting, since there are finitely many transmission signals in any time-frequency region, the codewords have finite length. Hence, the focus is on information capacity, rather than capacity.

II. PRELIMINARIES AND CHANNEL MODEL

A. Preliminaries

We must introduce several definitions before discussing the channel model. The modulation operator M_ω is $M_\omega f(t) = e^{2\pi i \omega t} f(t)$, and the translation operator T_x is $T_x f(t) = f(t - x)$. These operators have the commutation relation $T_x M_\omega = e^{-2\pi i x \omega} M_\omega T_x$ [14]. The Fourier transform of $f \in L^2(\mathbb{R})$, denoted \hat{f} , is

$$\hat{f}(\omega) = \int_{\mathbb{R}} f(t) e^{-2\pi i \omega t} dt,$$

and the corresponding operator is denoted \mathcal{F} . For $f \in L^2(\mathbb{R}^2)$, the operator \mathcal{F}_j , $j = 1, 2$, is given by

$$\mathcal{F}_j f(\cdot, \cdot) = \int_{\mathbb{R}} f(t_1, t_2) e^{-2\pi i \omega_j t_j} dt_j.$$

The cross-ambiguity function of f and $g \in L^2(\mathbb{R})$ is

$$\mathcal{A}(f, g)(x, \omega) = \int_{\mathbb{R}} f\left(t + \frac{x}{2}\right) \overline{g\left(t - \frac{x}{2}\right)} e^{-2\pi i t \omega} dt.$$

The cross-Wigner distribution of f and $g \in L^2(\mathbb{R})$ is

$$\mathcal{W}(f, g)(x, \omega) = \int_{\mathbb{R}} f\left(x + \frac{t}{2}\right) \overline{g\left(x - \frac{t}{2}\right)} e^{-2\pi i t \omega} dt.$$

Note: Unless otherwise specified, all integrals are over \mathbb{R} . Throughout we assume that $\alpha, \beta \geq 1$. C is used to denote a constant and can take different values in a calculation. For a real number x , let $(x)^+ = \max(0, x)$.

B. Channel Model

We model the channel as

$$r(t) = \int \int \hat{\sigma}(\omega, x) M_\omega T_x s(t) d\omega dx. \quad (3)$$

Therefore, the received signal is a weighted collection of modulated and delayed copies of the transmitted signal. However,

equation (3) can also be derived from the following time-varying convolution channel,

$$r(t) = \int h(t, t - \tau) s(\tau) d\tau, \quad (4)$$

where h is a time-varying impulse response. By defining $\sigma(t, \omega) = \mathcal{F}_2 h(t, \cdot)$ and using several Fourier transforms, see [14], we have

$$\int h(t, t - \tau) s(\tau) d\tau = \int \int \hat{\sigma}(\omega, x) M_\omega T_{-x} s(t) d\omega dx. \quad (5)$$

We therefore may equivalently view the channel as a Weyl pseudodifferential operator

$$\mathbf{L}_\sigma s(t) = \int \int \hat{\sigma}(\omega, x) e^{-\pi i x \omega} T_{-x} M_\omega s(t) d\omega dx. \quad (6)$$

The function $\hat{\sigma}$ is called the spreading function and σ the symbol of the operator \mathbf{L}_σ [14]. Our assumption is that the spreading function decays exponentially; that is, there exist constants $\alpha, \beta, C > 0$ such that

$$|\hat{\sigma}(\omega, x)| \leq C e^{-\beta|\omega| - \alpha|x|}. \quad (7)$$

Note first that equation (7) is more general than the common assumption of an *underspread* channel [15]. More importantly the approach to the time-varying channel indicated by the assumption in equation (7) does not rely on the notion of *coherence time*. We do not make any assumptions that the channel is constant during a coherence time period, as is discussed, for example in [16]. In fact, our channel may—as is physically sensible—be always smoothly evolving and never static.

Decay in the second variable of $\hat{\sigma}$ indicates that the time-varying impulse response decays exponentially. Since $\mathcal{F}_1^{-1}[\hat{\sigma}(\cdot, x)]$ gives the evolution of the impulse response over time, decay in the first variable of $\hat{\sigma}$ indicates that the higher order derivatives with respect to t of $h(t, \tau)$ decay rapidly; thus, $\sigma(t, \omega)$ evolves smoothly. A large β indicates that the channel is stable, or evolves smoothly, in time, whereas a large α indicates that the channel is stable, or varies smoothly, in frequency. Also, as $\beta \rightarrow \infty$ the channel approaches the time-invariant regime. The basic approach of our work is to design a system that can adapt to the parameters α and β to exploit either time or frequency stability. The next section establishes this framework.

III. SIGNALING SET AND FRAMES

A. Localized Signals

Our goal is to construct a set of signals that have “most” of their mass supported in the time-frequency region $[0, T] \times [-W, W]$ and are approximate eigenfunctions of the channel. Lemma (3.6) below shows that functions with appropriate time-frequency concentration approximately diagonalize the pseudodifferential operator channel given by (6) when the spreading function $\hat{\sigma}$ satisfies the decay condition (7).

The *uncertainty principle* is the foundation of any study concerning the time-frequency properties of a signaling set [17], [14]. The uncertainty principle entered into the discussion of capacity in an article of Wyner [18], in which he establishes

Shannon's capacity for the time-invariant channel using physically realizable signals.

Theorem 3.1: If $f \in L^2(\mathbb{R})$, then for all $a, b \in \mathbb{R}$

$$\frac{1}{4\pi} \|f\|_2^2 \leq \left(\int (x-a)^2 |f(x)|^2 dx \right)^{1/2} \times \left(\int (\omega-b)^2 |\hat{f}(\omega)|^2 d\omega \right)^{1/2},$$

and equality holds if and only if f is a (modulated, shifted, scaled) Gaussian.

The simplest consequence of this theorem is that a function in $L^2(\mathbb{R})$ cannot have compact time and frequency support. Therefore, the set of functions with time-frequency support exclusively contained in $[0, T] \times [-W, W]$ contains only the zero function; moreover, the set of functions with "most" of their mass supported in $[0, T] \times [-W, W]$, is not a closed subspace of $L^2(\mathbb{R})$, and, hence, does not have an orthonormal basis.

A second criterion for the signaling set is that the functions be linearly independent. If this is not the case, then even in the absence of noise a received signal does not uniquely determine the transmitted linear combination of signals. Thus, the problem is to construct a linearly independent set of functions with the appropriate time-frequency localization. One approach with a long history is to use prolate spheroidal wave functions, as for example Wyner does [18]. However, we prefer the Weyl-Heisenberg transmission signals, as these functions offer exponential decay in time and frequency and yield better capacity estimates.

There is a second consequence of the uncertainty principle that makes this task more difficult, namely the Balian-Low theorem [19].

Theorem 3.2: If $\{M_l T_k \phi\}_{k,l \in \mathbb{Z}}$ is a Riesz basis for $L^2(\mathbb{R})$, then either $x\phi(x) \notin L^2(\mathbb{R})$ or $\omega\hat{\phi}(\omega) \notin L^2(\mathbb{R})$.

That is, if the signaling set has the useful structural property of consisting of time-frequency shifts of an original function and is a linearly independent set, then it cannot be localized in *both* time and frequency. This theorem has often been overlooked in the literature (though a correct discussion may be found in [9]); some researchers have assumed the existence of an orthonormal basis (a type of Riesz basis) with time and frequency decay that violates the theorem. However, we may relax the condition that the closure of the signaling set span $L^2(\mathbb{R})$ and, in turn, obtain improved time-frequency localization. That is, we consider time-frequency shifts of an initial function that are orthonormal but the closure of which does not span $L^2(\mathbb{R})$. There are several steps necessary to prove that such a construction is possible; the first is to look at frames.

B. Weyl-Heisenberg Frames

Definition 3.3: [14] A subset $\{e_i\}_{i \in \Lambda}$ of a Hilbert space H is a *frame* for H if there exist constants A and B such that for all $f \in H$,

$$A \sum_{i \in \Lambda} |\langle f, e_i \rangle|^2 \leq \|f\|_2^2 \leq B \sum_{i \in \Lambda} |\langle f, e_i \rangle|^2.$$

A and B are called the *frame bounds* and a frame is *tight* if $A = B$. The *frame operator* \mathbf{S} is given by

$$\mathbf{S}f = \sum_{i \in \Lambda} \langle f, e_i \rangle e_i.$$

Note that \mathbf{S} is a positive operator.

Definition 3.4: If $(\phi, a, b) = \{M_{bl} T_{ak} \phi\}_{k,l \in \mathbb{Z}}$, $a, b \in \mathbb{R}^+$, is a frame, then it is called a *Gabor frame* or *Weyl-Heisenberg frame*. The *redundancy* is $\frac{1}{ab}$.

C. Signaling Set

In this section we design a signaling set $\Psi = \{\psi_{k,l}\}_{k,l \in \mathbb{Z}} \subset L^2(\mathbb{R})$ *adaptively* according to the time-frequency localization of the channel such that $\{\psi_{k,l}\}_{k,l \in \mathbb{Z}}$ are approximate eigenfunctions of \mathbf{L}_σ . (Approximate eigenfunction means that the operator applied to the function is approximately a scalar times the function; it is not necessarily related to approximate spectrum.) To communicate using the time-frequency region $[0, T] \times [-W, W]$ one selects a subset $\Psi_{\mathcal{I}} = \{\psi_{k,l}\}_{(k,l) \in \mathcal{I}} \subset \Psi$ approximately satisfying $\text{supp}(\psi_{k,l}) \subset [0, T]$ and $\text{supp}(\hat{\psi}_{k,l}) \subset [-W, W]$ for all $(k, l) \in \mathcal{I}$. The transmitted signals then have the form $s(t) = \sum_{(k,l) \in \mathcal{I}} c_{k,l} \psi_{k,l}$; and the model is

$$r(t) = \mathbf{L}_\sigma s(t) + n(t). \quad (8)$$

The operator \mathbf{L}_σ expressed as a matrix is

$$\mathbf{A}_{k,l,k',l'} = \langle \mathbf{L}_\sigma \psi_{k',l'}, \psi_{k,l} \rangle \quad (k, l) \in \mathcal{I}, \quad k', l' \in \mathbb{Z}. \quad (9)$$

The following proposition is critical: it states that a set $\Psi_{\mathcal{I}}$ with the desired properties exists. The signals in $\Psi_{\mathcal{I}}$ are approximate eigenfunctions of \mathbf{L}_σ , and we use them to address the eigenvalues of the matrix $\mathbf{A}^* \mathbf{A}$, which in turn determine the information capacity. Moreover, the proof of Proposition 3.5 may be viewed as a method for obtaining these signals.

Proposition 3.5: Let $g_s(t) = (2s)^{-1/4} e^{-\frac{\pi}{s} t^2}$, and set $\psi_s = \mathbf{S}^{-1/2} g_s$. Then $(\psi_s, \frac{a}{\rho}, \frac{b}{\rho}) = (\psi_s, \rho a, \rho b)$ ($ab = 1$ and $\rho > 1$) is an orthonormal system and there exist constants $C > 0$ and $0 < D < 1$ such that

$$|\psi_s(t)| \leq C e^{-D \frac{\pi}{s} |t|} \quad \forall t \in \mathbb{R}$$

$$|\widehat{\psi_s}(\omega)| \leq C e^{-D \pi s |\omega|} \quad \forall \omega \in \mathbb{R}.$$

Note here that the redundancy of $(\psi_s, \frac{a}{\rho}, \frac{b}{\rho})$ is $\frac{1}{\rho^2}$. The Weyl-Heisenberg set $(\psi_s, \rho a, \rho b)$ consists of time-frequency shifts of the window function ψ_s on the time-frequency lattice $a\rho\mathbb{Z} \times b\rho\mathbb{Z}$. Since $ab = 1$, the density of this lattice is $\frac{1}{\rho^2}$. Clearly, the closer ρ moves to 1, the more linearly independent functions get assigned to a rectangle of the time-frequency plane. Yet the Balian-Low theorem states that as $\rho \searrow 1$, the time-frequency localization is lost, which for this system means $D \rightarrow 0$. Additionally, the constant C increases dramatically. These effects are caused by the operator $\mathbf{S}^{-1/2}$, which becomes increasingly poorly conditioned as the density approaches 1. A detailed discussion of the density parameter ρ , however, is beyond the scope of this paper.

We give the proof of Proposition 3.5 here because it provides insight to the signaling system; the proofs of all

other theorems are given in Appendix A. *Proof:* A fundamental theorem due to Lyubarskii, Seip and Wallsten states that $(g_s, \frac{b}{\rho}, \frac{a}{\rho})$ is a frame for $L^2(\mathbb{R})$ if and only if $\frac{ab}{\rho^2} < 1$ [20], [21], [22]. By Theorem 5.1.6 and Corollary 7.3.2 in [14], $(\mathbf{S}^{-1/2}g_s, \frac{b}{\rho}, \frac{a}{\rho}) = (\psi_s, \frac{b}{\rho}, \frac{a}{\rho})$ is a tight frame for $L^2(\mathbb{R})$ with frame constant ρ^2 . By an essential result due to several authors [23], [24], [25], $(\psi, \frac{a}{\rho}, \frac{b}{\rho}) = (\psi, \rho b, \rho a)$ ($ab = 1$) is then an orthonormal set (which does not span $L^2(\mathbb{R})$). By Theorem 5 in [26], up to a factor $0 < D < 1$, the exponential decay of g_s and \hat{g}_s is preserved in ψ_s and $\hat{\psi}_s$. Finally, Theorem IV.2 in [27] implies that if ψ_1 is the window function for the orthonormal set based on the initial window g_1 , then ψ_s is the corresponding window function for g_s . ■

We use the parameters s, a and b to control the time and frequency localization of the signaling set and, hence, to adapt the signals to the channel. We will return to these parameters later.

D. Transmitter and Receiver Structure

We have two sets of functions: $\{M_{\rho bl}T_{\rho ak}\psi\}_{k,l \in \mathbb{Z}}$, is not linearly independent, but its closure spans $L^2(\mathbb{R})$; $\{M_{\rho bl}T_{\rho ak}\psi\}_{k,l \in \mathbb{Z}}$ is linearly independent, but its closure does not span $L^2(\mathbb{R})$. In order to preserve linear independence, the transmitter uses the first set to transmit data; however, the receiver will use the second set of functions to detect the signal. We set:

- $\psi^{nc} = \psi_s$
- $\psi^c = \frac{1}{\rho}\psi_s$
- $\psi_{k,l}^c = M_{\rho bl}T_{\rho ak}\psi^c$ ($ab = 1$)
- $\psi_{k,l}^{nc} = M_{\rho bl}T_{\rho ak}\psi^{nc}$ ($ab = 1$)
- $\mathbf{S}^c := \mathbf{S}^c f = \sum_{k,l \in \mathbb{Z}} \langle f, \psi_{k,l}^c \rangle \psi_{k,l}^c$

Here c stands for “complete” and nc stands for “not complete”.

The transmitter maps the information-bearing coefficients in \mathbb{C} to the orthonormal set $\{\psi_{k,l}^{nc}\}_{k,l \in \mathcal{I}}$ and transmits $s(t) = \sum_{(k,l) \in \mathcal{I}} c_{k,l} \psi_{k,l}^{nc}$, for the index set \mathcal{I} . It is possible that the channel moves signals from the non-spanning set $\{\psi_{k,l}^{nc}\}_{(k,l) \in \mathcal{I}}$ into its complement, and so if the receiver used the non-spanning set for detection, these signals would not be captured. Therefore, the receiver uses the spanning set $\{\psi_{k,l}^c\}_{k,l \in \mathbb{Z}}$ to detect the signal. We introduce the coefficient or analysis operator $\mathbf{C}^c : L^2(\mathbb{R}) \rightarrow l^2(\mathbb{Z}^2)$, given by

$$(\mathbf{C}^c f)_{k,l} = \langle f, \psi_{k,l}^c \rangle,$$

and its adjoint $\mathbf{C}^{c*} : l^2(\mathbb{Z}^2) \rightarrow L^2(\mathbb{R})$, given by

$$\mathbf{C}^{c*} \{a_{k,l}\}_{k,l \in \mathbb{Z}} = \sum_{k,l \in \mathbb{Z}} a_{k,l} \psi_{k,l}^c.$$

Note that $\mathbf{C}^{c*} \mathbf{C}^c = \mathbf{S}^c$. If the receiver functions $\{\psi_{k,l}^c\}_{k,l \in \mathbb{Z}}$ are not rescaled appropriately, for example if we use $\psi^c = \psi_s$, then receiver will amplify the received signal. By setting $\psi^c = \frac{1}{\rho}\psi_s$, $(\frac{1}{\rho}\psi_s, \frac{b}{\rho}, \frac{a}{\rho})$ is a tight frame with redundancy $\frac{1}{\rho^2}$, and $\|\mathbf{S}^c\| = \text{redundancy} \times \|\psi^c\| = 1$ [14], [19]. Therefore \mathbf{S}^c is an isometry on $L^2(\mathbb{R})$ and, hence, $n(t)$ and $\mathbf{S}^c n(t)$ have the same power spectral density. The channel may now be viewed as a map

$$\mathbf{C}^c \mathbf{L}_\sigma \mathbf{C}^{nc*} : l^2(\mathcal{I}) \rightarrow l^2(\mathbb{Z}^2). \quad (10)$$

This is expressed as a matrix as

$$\mathbf{A}_{klk'l'} = \langle \mathbf{L}_\sigma \psi_{k',l'}^{nc}, \psi_{k,l}^c \rangle. \quad (11)$$

Now the capacity of the channel restricted to the span of $\{\psi_{k',l'}^{nc}\}_{(k',l') \in \mathcal{I}}$ is given by

$$\sum_{(k,l) \in \mathcal{I}} \log \left(1 + \frac{\lambda_{kl}(\mathbf{A}^* \mathbf{A})}{\eta^2} \right). \quad (12)$$

The operator $\mathbf{A}^* \mathbf{A}$ is the matrix representation of

$$\mathbf{C}^{nc} \mathbf{L}_\sigma^* \mathbf{C}^{c*} \mathbf{C}^c \mathbf{L}_\sigma \mathbf{C}^{nc*} = \mathbf{C}^{nc} \mathbf{L}_\sigma^* \mathbf{S}^c \mathbf{L}_\sigma \mathbf{C}^{nc} \quad (13)$$

$$= \mathbf{C}^{nc} \mathbf{L}_\sigma^* \mathbf{L}_\sigma \mathbf{C}^{nc*}. \quad (14)$$

E. Approximate Diagonalization

The fact that the signals presented in Proposition 3.5 approximately diagonalize the operator \mathbf{L}_σ is the foundation for this entire work. This fact is made precise in the following lemma.

Lemma 3.6: Let $\psi = \mathbf{S}^{-1/2}(2s)^{-1/4}e^{-\frac{\pi}{s}t^2}$ and $\psi_{k,l}^c = M_{\rho bl}T_{\rho ak}\psi$ for positive constants a, b , $ab = 1$. Assume that the decay of $\hat{\sigma}$ is given by

$$|\hat{\sigma}(\omega, x)| \leq C e^{-\beta|\omega| - \alpha|x|}.$$

Then

$$\begin{aligned} |\langle \mathbf{L}_\sigma \psi_{k',l'}^{nc}, \psi_{k,l}^c \rangle| &\leq C(e^{-\beta br|\frac{1}{\rho^2}l-l'|} + e^{-\frac{\pi}{4s}Dbr|\frac{1}{\rho^2}l-l'|}) \\ &\quad \times (e^{-\alpha ar|\frac{1}{\rho^2}k-k'|} + e^{-\frac{\pi s D}{4}ar|\frac{1}{\rho^2}k-k'|}). \end{aligned}$$

IV. REGIONAL INFORMATION CAPACITY

A. Information Capacity with Receiver Channel Knowledge

The previous section established the existence of signals that approximately diagonalize the time-varying channel. Now we use the localization of these signals to a region of the time-frequency plane to determine the channel capacity specific to that region. Moreover, we relate the capacity to samples in that region of an “auto-convolved” form of the channel’s spreading function.

Let $R = [T_1, T_2] \times [W_1, W_2]$, $T_1 < T_2$, and $W_1 < W_2$, denote a region of the time-frequency plane. Without loss of generality we take $R = [0, T] \times [-W, W]$. Let the channel be given by $r = \mathbf{L}_\sigma s + n$, $n \sim \mathcal{N}(0, \eta^2)$, where $\hat{\sigma}$ satisfies the decay condition (7). We consider the following regime:

P1) $\psi = \mathbf{S}^{-1/2}2^{-1/4}(\frac{\alpha}{\beta})^{-1/2}e^{-\pi(\frac{\beta}{\alpha})^2 t^2}$.

P2) the signals are contained in the finite dimensional space

$$\begin{aligned} \Psi &= \text{span}\{\psi_{k,l}^{nc}\}_{k=0, l=-L}^{K,L} \\ &= \text{span}\{M_{\rho \frac{\alpha}{\beta} l} T_{\rho \frac{\beta}{\alpha} k} \psi\}_{k=0, l=-L}^{K,L}. \end{aligned}$$

P3) $K = \lfloor \frac{T}{\rho} \frac{\alpha}{\beta} \rfloor$ and $L = \lfloor \frac{2W}{\rho} \frac{\beta}{\alpha} \rfloor$.

Thus, while the following results are specific to the system just described, the system is sensible. This is evidenced by the fact that these signals are approximate eigenfunctions of \mathbf{L}_σ (Lemma 3.6), that the signals have structure and an algorithm for their construction (Proposition 3.5), and that they allow the capacity to be approximated by the sample values of an “auto-convolved” Weyl symbol (Theorem 4.1).

We assume that the receiver has perfect channel state information (CSIR), and that the transmitter knows only the decay parameters α and β and the rectangle $[0, T] \times [-W, W]$. Once the space Ψ given by property P2 is fixed, the matrix \mathbf{A} is determined, and the channel may be viewed as a standard matrix channel. Denoting the information capacity $\mathcal{IC}_{\sigma, R}$, we have

$$\mathcal{IC}_{\sigma, R} = \sum_{k=0, l=-L}^{K, L} \log \left(1 + \frac{\lambda_{kl}(\mathbf{A}^* \mathbf{A})}{\eta^2} \right).$$

Theorem 4.1 provides an estimate for this quantity in terms of a function \mathcal{S} that is derived from the time-varying impulse response h or, equivalently, the Weyl symbol σ .

Now we relate the eigenvalues of $\mathbf{A}^* \mathbf{A}$ to the Weyl symbol σ . The relationship between the spectral values and the sample values of the symbol is one of the difficulties of the time-varying channel and is a property that strongly distinguishes it from the time-invariant channel. For a time-invariant channel given by impulse response h_1 , the spectrum is given *exactly* by the values that \hat{h}_1 takes. In the time-varying setting, such a simple characterization does not exist. For example, a Weyl pseudodifferential operator may have all positive eigenvalues, yet its symbol may take negative values; an operator may also have a strictly positive symbol, yet not be a positive operator [28]. Thus, in many ways the spectral properties of time-varying convolution operators are much more complicated than their time-invariant counterparts. Yet, we are still able to approximate spectral values of the time-varying convolution operators by sample values of the appropriate symbol. The relationship that we prove here has been discussed by other authors [8], [11], but it had not yet been rigorously proved.

The composition of two Weyl pseudodifferential operators is given by $\mathbf{L}_\sigma \mathbf{L}_\tau = \mathbf{L}_{\sigma \sharp \tau}$, where $\sigma \sharp \tau = \mathcal{F}^{-1}(\hat{\sigma} \hat{\tau})$ is called the *twisted product* of σ and τ , and

$$(\hat{\sigma} \sharp \hat{\tau})(\omega, x) = \iint \hat{\sigma}(\omega', x') \hat{\tau}(\omega - \omega', x - x') e^{-\pi i(x\omega' - \omega x')} d\omega' dx$$

is called the *twisted convolution* of $\hat{\sigma}$ and $\hat{\tau}$. We have that $\sigma(t, \omega) = \mathcal{F}_2 h(t, \cdot)$, and we set $\mathcal{S} = \bar{\sigma} \sharp \sigma$, the Fourier transform of the “twisted auto-convolution” of $\hat{\sigma}$. The adjoint operator is $\mathbf{L}_\sigma^* = \mathbf{L}_{\bar{\sigma}}$ [14]. Therefore, since $\mathbf{L}_\sigma^* \mathbf{L}_\sigma$ is self adjoint, \mathcal{S} is a real-valued function. Lemma 1.2 shows that the appropriate samples of \mathcal{S} are approximately the diagonal entries of $\mathbf{A}^* \mathbf{A}$, which are real and positive. However, if $(\mathbf{A}^* \mathbf{A})_{klkl}$ is small, the corresponding sample of \mathcal{S} may, in fact, be negative. We, therefore, work with $\mathcal{S}^+(x, \omega) = (\mathcal{S}(x, \omega))^+$ so that negative values do not enter into the log function.

The channel capacity is then approximately given by the samples of \mathcal{S} , analogously to samples of $|\hat{h}_1|^2$ giving the capacity of a time-invariant channel with impulse response h_1 . Moreover, these samples are all taken in the region of the time-frequency plane in which the signaling functions are localized. We, therefore, have capacity in terms of two physical properties: the time-varying transfer function and the time-frequency region.

Theorem 4.1 (Information Capacity for CSIR): Let R be a closed rectangular region of the time-frequency plane. Assume the channel is given by $r = \mathbf{L}_\sigma s + n$, $n \sim \mathcal{N}(0, \eta^2)$, that

$\hat{\sigma}$ satisfies the decay condition (7), and that the receiver has perfect channel knowledge, while the transmitter knows only α , β and R . Also assume the signaling set is given according to properties P1-P3 above. Set $\mathcal{S} = \bar{\sigma} \sharp \sigma$ and $\mathcal{S}^+(x, \omega) = (\mathcal{S}(x, \omega))^+$. Then

$$\left| \mathcal{IC}_{\sigma, R} - \sum_{k=0, l=-L}^{K, L} \log \left(1 + \frac{\mathcal{S}^+(\rho_\alpha^\beta k, \rho_\beta^\alpha l)}{\eta^2} \right) \right| \leq 2KL \log(1 + \mathcal{O}(e^{-\frac{\beta}{4}(\beta+\alpha)} + \frac{1}{(\alpha\beta D)^2}))$$

B. Information Capacity with Transmitter Channel Knowledge

Now we treat the same channel as discussed in the previous section under the assumption that both the transmitter and receiver have perfect channel knowledge (CSIT). The space Ψ given by P2 as in the previous section is identical, and again the channel may then be viewed as a standard matrix channel. Since the transmitter and receiver both have perfect channel knowledge, the transmitter will use the eigenvectors of $\mathbf{A}^* \mathbf{A}$ for transmission and allocate power according to water-filling [4]. We denote the information capacity for the CSIT regime by $\mathcal{IC}_{\sigma, R}^w$, and we have

$$\mathcal{IC}_{\sigma, R}^w = \sum_{k=0, l=-L}^{K, L} \log \left(1 + \frac{P_{kl} \lambda_{kl}(\mathbf{A}^* \mathbf{A})}{\eta^2} \right), \quad (15)$$

where $\{P_{kl}\}_{k=0, l=-L}^{K, L}$ denotes the power allocated to the eigenvectors and $\sum_{k=0, l=-L}^{K, L} P_{kl} = P$ is the total power available.

Theorem 4.1 shows that the eigenvalues of $\mathbf{A}^* \mathbf{A}$ are approximated by the sample values of \mathcal{S} (this is stated explicitly as Corollary 1.3). Therefore, rather than computing first the matrix $\mathbf{A}^* \mathbf{A}$ and its eigenvalues, we can approximate these eigenvalues by just sampling the symbol \mathcal{S} . In order to approximate $\mathcal{IC}_{\sigma, R}^w$ we require approximations for the power allocations $\{P_{kl}\}_{k=0, l=-L}^{K, L}$. Again, rather than computing these from the eigenvalues of $\mathbf{A}^* \mathbf{A}$, we allocate power based on the approximate eigenvalues $\{\mathcal{S}^+(\rho_\alpha^\beta k, \rho_\beta^\alpha l)\}_{k=0, l=-L}^{K, L}$.

Theorem 4.2 (Information Capacity for CSIT): Let R be a closed rectangular region of the time-frequency plane. Assume the channel is given by $r = \mathbf{L}_\sigma s + n$, $n \sim \mathcal{N}(0, \eta^2)$, that $\hat{\sigma}$ satisfies the decay condition (7), and that both the transmitter and receiver have perfect channel knowledge. Also assume the signaling set is given according to properties P1-P3 above. Set $\mathcal{S} = \bar{\sigma} \sharp \sigma$ and $\mathcal{S}^+(x, \omega) = (\mathcal{S}(x, \omega))^+$. Let P be the total power available to the transmitter. Then

$$\left| \mathcal{IC}_{\sigma, R}^w - \sum_{k=0, l=-L}^{K, L} \log \left(1 + \frac{P_{kl}^S \mathcal{S}^+(\rho_\alpha^\beta k, \rho_\beta^\alpha l)}{\eta^2} \right) \right| \leq 2KL \log(1 + 2KL \cdot \mathcal{O}(e^{-\frac{\beta}{4}(\beta+\alpha)} + \frac{1}{(\alpha\beta D)^2})),$$

where, $\{P_{kl}^S\}_{k=0, l=-L}^{K, L}$, $\sum_{k=0, l=-L}^{K, L} P_{kl}^S = P$, denotes the water-filling allocation based on the approximate eigenvalues $\{\mathcal{S}^+(\rho_\alpha^\beta k, \rho_\beta^\alpha l)\}_{k=0, l=-L}^{K, L}$.

V. RECOVERY OF TIME-INVARIANT CAPACITY

In the previous section we assumed the channel spreading function satisfied

$$|\hat{\sigma}(\omega, x)| \leq C e^{-\beta|\omega| - \alpha|x|}.$$

We now consider a sequence of time-varying channels approaching the time-invariant channel. We assume the sequence of Weyl symbols has the following properties:

P4) $|\hat{\sigma}_n(\omega, x)| \leq C_n e^{-\beta_n|\omega| - \alpha|x|}$

Note that α is fixed for all $n \in \mathbb{N}$.

P5) there exist positive constants M and m such that

$$M \geq \int \int |\hat{\sigma}(\omega, x)| d\omega dx \geq m \quad \forall n \in \mathbb{N}.$$

P6) $\beta_n > \beta_{n'}$ for $n > n'$ and $\lim_{n \rightarrow \infty} \beta_n = \infty$.

If the sequence $\{\hat{\sigma}_n(\omega, x)\}_{n \in \mathbb{N}}$ satisfies these properties, then $\hat{\sigma}_n(\omega, x) \rightarrow \delta(\omega)h(x)$, and therefore $\sigma_n(x, \omega) \rightarrow \hat{h}(\omega)$. The generalized eigenvectors of a time-invariant channel are pure frequencies. So as the channel becomes more stable, the eigenvectors approach pure frequencies. One then exploits this stability by transmitting signals closer to pure frequencies, which means transmitting longer signals. We assume that the channel is band-limited to $[-W, W]$.

Theorem 5.1 (Time-Invariant Capacity for CSIR): Let $\{\sigma_n\}_{n \in \mathbb{N}}$ be a sequence of Weyl symbols converging to the linear time-invariant channel given by the impulse response $h(x)$ according to properties P4-P6. Set $R_n = [0, \frac{\beta_n}{\alpha}] \times [-W, W]$, and let the noise and signaling system be as in Theorem 4.1. Then

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{1}{|R_n|} \sum_{k=0, l=-L}^{K, L} \log \left(1 + \frac{\mathcal{S}^+(\rho_{\alpha}^{\beta} k, \rho_{\beta}^{\alpha} l)}{\eta^2} \right) \\ = \frac{1}{2\rho W} \int_{-W}^W \log \left(1 + \frac{|\hat{h}(\omega)|^2}{\eta^2} \right) d\omega. \end{aligned}$$

Theorem 5.2 (Time-Invariant Capacity for CSIT):

We make the same hypotheses and definitions as for Theorem 5.1, except that the transmitter now also has perfect channel knowledge and total power resources P_{Total} . Let $\{P_{kl}^{S_n}\}_{k=0, l=-L}^{K, L}$ denote the water-filling power allocations based on the approximate eigenvalues $\mathcal{S}_n^+(\rho_{\alpha}^{\beta} k, \rho_{\beta}^{\alpha} l)$. Then

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{1}{|R_n|} \sum_{k=0, l=-L}^{K, L} \log \left(1 + \frac{P_{kl}^{S_n} \mathcal{S}^+(\rho_{\alpha}^{\beta} k, \rho_{\beta}^{\alpha} l)}{\eta^2} \right) \\ = \frac{1}{2\rho W} \int_{-W}^W \log \left(1 + \frac{P(\omega) |\hat{h}(\omega)|^2}{\eta^2} \right) d\omega, \end{aligned}$$

where $P(\omega)$ is the power allocation determined by water-filling, as in the classical time-invariant case [4], and $\int_{-W}^W P(\omega) d\omega = P_{Total}$.

APPENDIX I PROOFS OF THEOREMS

Proof: [Proof of Lemma 3.6] The following two essential identities hold for Weyl pseudodifferential operators [14]:

$$\langle \mathbf{L}_{\sigma} f, g \rangle = \langle \sigma, \mathcal{W}(g, f) \rangle \quad (16)$$

$$|\langle \mathbf{L}_{\sigma} T_u M_{\eta} f, T_v M_{\gamma} g \rangle| = |(\hat{\sigma} * \mathcal{A}(f, g))(u - v, \eta - \gamma)|. \quad (17)$$

We define the function ϕ for $c_1, c_2 \geq 1$ and will use the following bound:

$$\begin{aligned} \phi(c_1, c_2, X) &= \int e^{-c_1|y|} e^{-c_2|X-y|} dy \\ &= \frac{1}{c_1 + c_2} (e^{-c_1|X|} + e^{-c_2|X|}) \\ &\quad + \frac{1}{2 - c_1} (e^{-c_1|X|} - e^{-c_2|X|}) \\ &\leq C e^{e^{-c_1|X|} + e^{-c_2|X|}}. \end{aligned}$$

The system is given by $\psi_s = \mathbf{S}^{-1/2} g_s$, where $g_s(t) = (2s)^{-1/4} e^{-\frac{\pi}{s} t^2}$, and by Proposition (3.5)

$$|\psi_s(t)| \leq C e^{-\frac{\pi}{s} D |t|}$$

$$|\hat{\psi}_s(\omega)| \leq C e^{-\pi s D |\omega|}.$$

Lemma 2.1 implies

$$|\mathcal{A}(\psi, \psi)(\omega, x)| \leq C e^{-\frac{\pi}{4s} D |\omega| - \frac{\pi s D}{4} |x|}.$$

$$\begin{aligned} |\mathbf{A}_{k, l, k', l'}| &= |\langle \mathbf{L}_{\sigma} \psi_{k' l'}, \psi_{k l} \rangle| \\ &= |\langle \mathbf{L}_{\sigma} M_{\rho b l'} T_{\rho a k'} \psi, M_{\frac{1}{\rho} b l} T_{\frac{1}{\rho} a k} \psi \rangle| \\ &= |(\hat{\sigma} * \mathcal{A}(\psi, \psi))(a(\frac{1}{\rho} k - \rho k'), b(\frac{1}{\rho} l - \rho l'))| \\ &= \left| \iint \hat{\sigma}(\omega, x) \mathcal{A}(\psi, \psi) \right. \\ &\quad \left. (a(\frac{1}{\rho} k - \rho k') - x, b(\frac{1}{\rho} l - \rho l') - \omega) d\omega dx \right| \\ &\leq C \iint e^{-\beta|\omega| - \alpha|x|} \\ &\quad e^{-\frac{\pi}{4s} D |a(\frac{1}{\rho} k - \rho k') - x| - \frac{\pi s D}{4} |b(\frac{1}{\rho} l - \rho l') - \omega|} d\omega dx \\ &= C \int e^{-\beta|\omega| - \frac{\pi}{4s} D |b(\frac{1}{\rho} l - \rho l') - \omega|} d\omega \\ &\quad \times \int e^{-\alpha|x| - \frac{\pi s D}{4} |a(\frac{1}{\rho} k - \rho k') - x|} dx \\ &= C \phi(\beta, \frac{\pi}{4s} D, b(\frac{1}{\rho} l - \rho l')) \phi(\alpha, \frac{\pi s D}{4}, a(\frac{1}{\rho} k - \rho k')) \\ &\leq C (e^{-\beta b r |\frac{1}{\rho^2} l - l'|} + e^{-\frac{\pi}{4s} D b \rho |\frac{1}{\rho^2} l - l'|}) \\ &\quad \times (e^{-\alpha a r |\frac{1}{\rho^2} k - k'|} + e^{-\frac{\pi s D}{4} a \rho |\frac{1}{\rho^2} k - k'|}). \end{aligned}$$

For Theorem 4.1, we first prove the following lemmas. ■

Lemma 1.1: Assume the hypotheses of Theorem 4.1. Then

$$\begin{aligned} |\mathcal{I}\mathcal{C}_{\sigma, R}^w - \log(1 + P_{kl}(\mathbf{A}^* \mathbf{A})_{klkl})| \\ \leq 2KL \log \left(1 + \mathcal{O}(e^{-\frac{\rho}{4}(\beta + \alpha)}) \right) \end{aligned}$$

Proof:

$$\begin{aligned} |(\mathbf{A}^* \mathbf{A})_{klk' l'}| \\ = \left| \sum_{j, j' \in \mathbb{Z}} \mathbf{A}_{j j' k l} \mathbf{A}_{j j' k' l'} \right| \end{aligned}$$

$$\begin{aligned}
&\leq \sum_{j,j' \in \mathbb{Z}} |\mathbf{A}_{jj'kl}| |\mathbf{A}_{jj'k'l'}| \\
&= C \sum_{j \in \mathbb{Z}} \{ (e^{-\beta b \rho |l - \frac{1}{\rho^2} j'|} + e^{-\frac{\pi}{4s} D b \rho |l - \frac{1}{\rho^2} j'|}) \\
&\quad \times (e^{-\beta b \rho |\frac{1}{\rho^2} j' - l'|} + e^{-\frac{\pi}{4s} D b \rho |\frac{1}{\rho^2} j' - l'|}) \} \\
&\quad \times \sum_{j \in \mathbb{Z}} \{ (e^{-\alpha a \rho |k - \frac{1}{\rho^2} j|} + e^{-\frac{\pi s D}{4} a \rho |k - \frac{1}{\rho^2} j|}) \\
&\quad \times (e^{-\alpha a \rho |\frac{1}{\rho^2} j - k'|} + e^{-\frac{\pi s D}{4} a \rho |\frac{1}{\rho^2} j - k'|}) \} \\
&= C \sum_{j \in \mathbb{Z}} \{ e^{-\beta b \rho (|l - \frac{1}{\rho^2} j'| + |\frac{1}{\rho^2} j' - l'|)} \\
&\quad + 2e^{-\beta b \rho |l - \frac{1}{\rho^2} j'| - \frac{\pi}{4s} D b \rho |\frac{1}{\rho^2} j' - l'|} \\
&\quad + e^{-\frac{\pi}{4s} D b \rho (|l - \frac{1}{\rho^2} j'| + |\frac{1}{\rho^2} j' - l'|)} \} \\
&\quad \times \sum_{j \in \mathbb{Z}} \{ e^{-\alpha a \rho (|k - \frac{1}{\rho^2} j| + |\frac{1}{\rho^2} j - k'|)} \\
&\quad + 2e^{-\alpha a \rho |k - \frac{1}{\rho^2} j| - \frac{\pi s D}{4} a \rho |\frac{1}{\rho^2} j - k'|} \\
&\quad + e^{-\frac{\pi s D}{4} a \rho (|k - \frac{1}{\rho^2} j| + |\frac{1}{\rho^2} j - k'|)} \} \\
&\leq C (e^{-\beta b \rho \frac{|l-l'|}{2}} + e^{-\frac{\pi}{8s} D b \rho |l-l'|}) \\
&\quad \times (e^{-\alpha a \rho \frac{|k-k'|}{2}} + e^{-\frac{\pi s D}{8} a \rho |k-k'|})
\end{aligned}$$

where we have used Lemmas 3.6 and 2.2. Next

$$\begin{aligned}
&\sum_{\substack{k=-K, \dots, K, k \neq k' \\ l=-L, \dots, L, l \neq l'}} |(\mathbf{A}^* \mathbf{A})_{klk'l'}| \\
&\leq C \sum_{k=-K, \dots, K, k \neq k'} (e^{-\alpha a \rho \frac{|k-k'|}{2}} + e^{-\frac{\pi s D}{8} a \rho |k-k'|}) \\
&\quad \times \sum_{l=-L, \dots, L, l \neq l'} (e^{-\beta b \rho \frac{|l-l'|}{2}} + e^{-\frac{\pi}{8s} D b \rho |l-l'|}) \\
&= \mathcal{O} \left(\left(\frac{e^{-\alpha a \rho/2}}{1 - e^{-\alpha a \rho/2}} + \frac{e^{-\frac{\pi s D}{8} a \rho}}{1 - e^{-\frac{\pi s D}{8} a \rho}} \right) \right. \\
&\quad \left. \times \left(\frac{e^{-\beta b \rho/2}}{1 - e^{-\beta b \rho/2}} + \frac{e^{-\frac{\pi}{8s} D b \rho}}{1 - e^{-\frac{\pi}{8s} D b \rho}} \right) \right). \quad (19)
\end{aligned}$$

Now we set

$$a = \frac{\beta}{\alpha}, \quad b = \frac{\alpha}{\beta}, \quad s = \frac{a}{b} = \left(\frac{\beta}{\alpha} \right)^2.$$

Then

$$\alpha a = \beta, \quad s a = \left(\frac{\beta}{\alpha} \right)^3, \quad \beta b = \alpha \quad \text{and} \quad \frac{b}{s} = \left(\frac{\alpha}{\beta} \right)^3,$$

and equation (19) becomes

$$\begin{aligned}
&= \mathcal{O} \left(\left(\frac{e^{-\beta \rho/2}}{1 - e^{-\beta \rho/2}} + \frac{e^{-(\frac{\beta}{\alpha})^3 \pi D \rho/8}}{1 - e^{-(\frac{\beta}{\alpha})^3 \pi D \rho/8}} \right) \right. \\
&\quad \left. \times \left(\frac{e^{-\alpha \rho/2}}{1 - e^{-\alpha \rho/2}} + \frac{e^{-(\frac{\alpha}{\beta})^3 \pi D \rho/8}}{1 - e^{-(\frac{\alpha}{\beta})^3 \pi D \rho/8}} \right) \right) \\
&= \mathcal{O} \left(\frac{e^{-\frac{\rho}{2}(\beta+\alpha)}}{(1 - e^{-(\frac{\beta}{\alpha})^3 \pi D \rho/8})(1 - e^{-(\frac{\alpha}{\beta})^3 \pi D \rho/8})} \right) \\
&= \mathcal{O} \left(e^{-\frac{\rho}{4}(\beta+\alpha)} \right). \quad (20)
\end{aligned}$$

Using Lemma 2.6,

$$\begin{aligned}
&|\mathcal{I}C_{\sigma,R}^w - \log(1 + P_{kl}(\mathbf{A}^* \mathbf{A})_{klkl})| \\
&\leq 2KL \log \left(1 + \mathcal{O}(e^{-\frac{\rho}{4}(\beta+\alpha)}) \right)
\end{aligned}$$

Lemma 1.2: Let $\mathcal{S} = \overline{\sigma} \# \sigma$. Then

$$|(\mathbf{A}^* \mathbf{A})_{k,l,k,l} - \mathcal{S}(\rho \frac{\beta}{\alpha} k, \rho \frac{\alpha}{\beta} l)| = \mathcal{O} \left(\frac{1}{(\alpha \beta D)^2} \right).$$

Proof: We first look at $(\mathbf{A}^* \mathbf{A})_{klkl}$. The diagonal entries of $\mathbf{A}^* \mathbf{A}$ are

$$\begin{aligned}
(\mathbf{A}^* \mathbf{A})_{klkl} &= \sum_{k',l' \in \mathbb{Z}^2} |\langle \mathbf{L}_\sigma \psi_{k,l}^{nc}, \psi_{k',l'}^c \rangle|^2 \\
&= \|\mathbf{L}_\sigma \psi_{k,l}^{nc}\|_2^2,
\end{aligned}$$

since $(\psi, \frac{1}{\rho} a, \frac{1}{\rho} b)$ is tight Gabor frame (Proposition (3.5)).

$$\begin{aligned}
\|\mathbf{L}_\sigma \psi_{k,l}^{nc}\|_2^2 &= \langle \mathbf{L}_\sigma \psi_{k,l}^{nc}, \mathbf{L}_\sigma \psi_{k,l}^{nc} \rangle \\
&= \langle \overline{\sigma} \# \sigma, \mathcal{W}(\psi_{k,l}^{nc}, \psi_{k,l}^{nc}) \rangle \\
&= \int_{\mathbb{R}^2} \mathcal{S}(x, \omega)
\end{aligned}$$

$$\mathcal{W}(\psi_{k,l}^{nc}, \psi_{k,l}^{nc})(x - \rho \frac{\beta}{\alpha} k, \omega - \rho \frac{\alpha}{\beta} l) d\omega dx$$

By the Riemann-Lebesgue Lemma,

$$\begin{aligned}
\|\mathcal{S}'\|_\infty &\leq \iint |\hat{\mathcal{S}}(\omega, x)| d\omega dx \\
&\leq 9C^2 \iint \frac{1}{\alpha \beta} e^{-\frac{\beta}{2}|\omega| - \frac{\alpha}{2}|x|} d\omega dx \\
&= \frac{9 \cdot 2^4 C^2}{(\alpha \beta)^2}.
\end{aligned}$$

We use Lemma 2.1 and the fact that $\iint W(\psi, \psi)(\omega, x) d\omega dx = \|\psi\|_2^2 = 1$ [14].

$$\begin{aligned}
&\| \|\mathbf{L}_\sigma \psi_{k,l}^{nc}\|_2^2 - \mathcal{S}(\rho \frac{\beta}{\alpha} k, \rho \frac{\alpha}{\beta} l) \| \\
&= \left| \int_{\mathbb{R}^2} \mathcal{S}(x, \omega) \mathcal{W}(\psi, \psi)(x - \rho \frac{\beta}{\alpha} k, \omega - \rho \frac{\alpha}{\beta} l) d\omega dx \right. \\
&\quad \left. - \mathcal{S}(\rho \frac{\alpha}{\beta} l, \rho \frac{\beta}{\alpha} k) \right| \\
&= \left| \int_{\mathbb{R}^2} \mathcal{S}(x + \rho \frac{\beta}{\alpha} k, \omega + \rho \frac{\alpha}{\beta} l) \mathcal{W}(\psi, \psi)(x, \omega) d\omega dx \right. \\
&\quad \left. - \mathcal{S}(\rho \frac{\alpha}{\beta} l, \rho \frac{\beta}{\alpha} k) \right| \\
&= \left| \int_{\mathbb{R}^2} [\mathcal{S}(x + \rho \frac{\beta}{\alpha} k, \omega + \rho \frac{\alpha}{\beta} l) - \mathcal{S}(\rho \frac{\alpha}{\beta} l, \rho \frac{\beta}{\alpha} k)] \right. \\
&\quad \left. \mathcal{W}(\psi, \psi)(x, \omega) d\omega dx \right| \\
&\leq \|\mathcal{S}'\|_\infty \int_{\mathbb{R}^2} (|x| + |\omega|) \mathcal{W}(\psi, \psi)(x, \omega) d\omega dx \\
&\leq C \frac{1}{(\alpha \beta)^2} \int_{\mathbb{R}^2} (|x| + |\omega|) e^{-\frac{\pi s D}{4} |x| - \frac{\pi}{4s} D |\omega|} d\omega dx \\
&= C \frac{1}{(\alpha \beta D)^2}
\end{aligned}$$

These two bounds prove the lemma. ■

Lemma 1.3: Let $\mathcal{S} = \overline{\sigma}_n^\# \sigma$ and $S^+(x, \omega) = (S(x, \omega))^+$. Then

$$\begin{aligned} & \left| \log(1 + \lambda_{k,l}(\mathbf{A}^* \mathbf{A})) - \log(1 + \mathcal{S}^+(\rho \frac{\beta}{\alpha} k, \rho \frac{\alpha}{\beta} l)) \right| \\ &= \log \left(1 + \mathcal{O} \left(e^{-\frac{\rho}{4}(\beta+\alpha)} + \frac{1}{(\alpha\beta D)^2} \right) \right) \end{aligned}$$

Proof: Using Lemmas 1.1 and 1.2,

$$\begin{aligned} & \left| \log(1 + \lambda_{k,l}(\mathbf{A}^* \mathbf{A})) - \log(1 + \mathcal{S}^+(\rho \frac{\beta}{\alpha} k, \rho \frac{\alpha}{\beta} l)) \right| \\ & \leq \left| \log(1 + \lambda_{k,l}(\mathbf{A}^* \mathbf{A})) - \log(1 + (\mathbf{A}^* \mathbf{A})_{klkl}) \right| \\ & + \left| \log(1 + (\mathbf{A}^* \mathbf{A})_{klkl}) - \log(1 + \mathcal{S}^+(\rho \frac{\beta}{\alpha} k, \rho \frac{\alpha}{\beta} l)) \right| \\ &= \log \left(1 + \mathcal{O} \left(e^{-\frac{\rho}{4}(\beta+\alpha)} + \frac{1}{(\alpha\beta D)^2} \right) \right) \end{aligned}$$

Proof: [Proof of Theorem 4.1] The proof follows from Lemmas 1.1 and 1.3

Proof: [Proof of Theorem 4.2] Applying Lemma 2.8 to Lemma 1.3 yields the result.

Proof: [Proof of Theorem 5.1] We set $K = \lfloor \frac{1}{\rho} \rfloor = 0$ and $L = \lfloor \frac{2W}{\rho} \frac{\beta_n}{\alpha} \rfloor$. Also, set $\mathcal{S}_n = \overline{\sigma}_n^\# \sigma_n$ and $\{\mathcal{S}_n^+(x, \omega) = (S(x, \omega))^+\}$. Theorem 4.1 yields the following:

$$\begin{aligned} & \lim_{n \rightarrow \infty} \mathcal{I} \mathcal{C}_{\sigma_n, R_n} \\ &= \lim_{n \rightarrow \infty} \frac{1}{\frac{\beta_n}{\alpha} 2W \rho^2} \sum_{k=0}^0 \sum_{l=-L_n}^{L_n} \log \left(1 + \frac{\mathcal{S}_n^+(\rho \frac{\beta_n}{\alpha} k, \rho \frac{\alpha}{\beta_n} l)}{\eta^2} \right) \\ &= \lim_{n \rightarrow \infty} \frac{1}{2W \rho^2} \frac{\alpha}{\beta_n} \sum_{l=-\frac{1}{\rho} \frac{\beta_n}{\alpha} W}^{\frac{1}{\rho} \frac{\beta_n}{\alpha} W} \log \left(1 + \frac{\mathcal{S}_n^+(0, \rho \frac{\alpha}{\beta_n} l)}{\eta^2} \right) \\ &= \lim_{n \rightarrow \infty} \frac{1}{\rho^2} \frac{1}{2W} \frac{\alpha}{\beta_n} \sum_{l=-\frac{1}{\rho} \frac{\beta_n}{\alpha} W}^{\frac{1}{\rho} \frac{\beta_n}{\alpha} W} \log \left(1 + \frac{|\hat{h}(\rho \frac{\alpha}{\beta_n} l)|^2}{\eta^2} \right) \\ &= \frac{1}{2\rho W} \int_{-W}^W \log \left(1 + \frac{|\hat{h}(\omega)|^2}{\eta^2} \right) d\omega \end{aligned}$$

Therefore, up to the redundancy, we recover the normalized time-invariant information capacity:

$$\lim_{n \rightarrow \infty} \mathcal{I}_{\sigma_n, R_n} = \frac{1}{\rho} \frac{1}{2W} \int_{-W}^W \log \left(1 + \frac{|\hat{h}(\omega)|^2}{\eta^2} \right) d\omega.$$

Proof: [Proof of Theorem 5.2] This proof relies on Theorem 4.2 in the same way that the proof of Theorem 5.1 relies on Theorem 4.1. The only difference is that one must consider the convergence of $\{P_{kl}^{S_n}\}_{k=0, l=-L}^{K, L}$ to $P(\omega)$. This convergence is given by the fact that as $\beta_n \rightarrow \infty$, the approximate eigenvalues converge to the exact eigenvalues, Lemma 1.3. Consequently, the estimated power allocation also converges to the optimal power allocation, Lemma 15. The optimal power allocation for each σ_n then converges to $P(\omega)$ as $n \rightarrow \infty$ since the values $\mathcal{S}^+(\rho \frac{\beta}{\alpha} k, \rho \frac{\alpha}{\beta} l)$ converge to samples of $|\hat{h}(\omega)|^2$.

We note that the factor $2KL$ inside the log in the statement of Theorem 4.2 is now $2W \frac{\beta_n}{\alpha}$. Therefore, the error decay given in Theorem 4.2 will only be of the order $\mathcal{O}(\beta_n^{-1})$. ■

APPENDIX II LEMMAS USED

Lemma 2.1: If $|\psi(x)| \leq C e^{-c_1|x|}$ and $|\hat{\psi}(\omega)| \leq C e^{-c_2|\omega|}$ for $c_1, c_2 > 0$, then

$$|W(\psi, \psi)(x, \omega)| \leq C^2 e^{-\frac{1}{4}(c_1|x|+c_2|\omega|)}$$

and

$$|A(\psi, \psi)(x, \omega)| \leq C^2 e^{-\frac{1}{4}(c_1|x|+c_2|\omega|)}.$$

Proof: The proof is contained in the proof of Theorem 2.4 in [29], when one views both distributions as short-time Fourier transforms, as explained in [14]. ■

Lemma 2.2: For α and $p > 0$,

$$\sum_{j \in \mathbb{Z}} e^{-\alpha(|k-pj|+|pj-k'|)} \leq C e^{-\frac{\alpha}{2}|k-k'|},$$

and for $\alpha, \beta, p > 0$, $\alpha \neq \beta$,

$$\sum_{j \in \mathbb{Z}} e^{-\alpha|k-pj|-\beta|pj-k'|} \leq C(e^{-\beta \frac{|k-k'|}{2}} + e^{-\alpha \frac{|k-k'|}{2}}).$$

Proof: w.l.o.g., $k < k'$.

- for $pj \in [k, k']$, $|k-pj| + |k'-pj| = |k-k'|$
- for $pj > k'$, $pj = k' + b$, $b > 0$, $|k-pj| + |k'-pj| = |k-(k'+b)| + |b| = |k-k'| + 2b$
- for $pj < k$, $pj = k - b$, $b > 0$, $|k-pj| + |k'-pj| = |b| + |k'-(k-b)| = |k-k'| + 2b$

$$\begin{aligned} & \sum_{j \in \mathbb{Z}^2} e^{-\alpha(|k-j|+|k'-j|)} \\ &= \lfloor (1+p|k-k'|) \rfloor e^{-\alpha|k-k'|} + 2 \sum_{j=1}^{\infty} e^{-2\alpha pj} e^{-\alpha|k-k'|} \\ &= \lfloor (1+p|k-k'|) \rfloor e^{-\alpha|k-k'|} + 2e^{-\alpha|k-k'|} \left(\frac{1}{1-e^{-2p\alpha}} - 1 \right) \\ &= \lfloor (1+p|k-k'|) \rfloor e^{-\alpha|k-k'|} + 2e^{-\alpha|k-k'|} \frac{e^{-2p\alpha}}{1-e^{-2p\alpha}} \\ &\leq C e^{-\frac{\alpha}{2}|k-k'|}. \end{aligned}$$

Again, w.l.o.g., $k < k'$.

$$\begin{aligned} \sum_{j \geq \lfloor \frac{k'}{p} \rfloor} e^{-\alpha|k-pj|-\beta|pj-k'|} &\leq e^{-\alpha|k-k'|} \sum_{j=0}^{\infty} e^{-(\alpha+\beta)pj} \\ &= C e^{-\alpha|k-k'|}. \end{aligned} \quad (21)$$

Similarly,

$$\sum_{j \leq \lfloor \frac{k}{p} \rfloor} e^{-\alpha|k-pj|-\beta|pj-k'|} \leq C e^{-\beta|k-k'|}. \quad (22)$$

Set $N = \lceil \frac{|k-k'|}{2p} \rceil$.

$$\sum_{\lfloor \frac{k}{p} \rfloor < j < \lfloor \frac{k'}{p} \rfloor} e^{-\alpha|k-pj|-\beta|pj-k'|}$$

$$\begin{aligned}
&\leq \sum_{j=1}^N (e^{-\alpha p j} e^{-\beta(|k-k'|-p j)} + e^{-\alpha(|k-k'|-p j)} e^{-\beta p j}) \\
&= e^{-\beta|k-k'|/2} \sum_{j=1}^N e^{-\alpha p j} e^{-\beta(\frac{|k-k'|}{2}-p j)} \\
&\quad + e^{-\alpha|k-k'|/2} \sum_{j=1}^N e^{-\beta p j} e^{-\alpha(\frac{|k-k'|}{2}-p j)} \\
&= C(e^{-\beta\frac{|k-k'|}{2}} + e^{-\alpha\frac{|k-k'|}{2}}) \tag{23}
\end{aligned}$$

Finally, (21) + (22) + (23) $\leq C(e^{-\beta\frac{|k-k'|}{2}} + e^{-\alpha\frac{|k-k'|}{2}})$. ■

Definition 2.3: [30] Let $x, y \in \mathbb{R}^n$ satisfy $\sum_{i=1}^n x_i = \sum_{i=1}^n y_i$. Let x' and y' be the vectors given by reordering x and y so that $x'_i \geq x'_j$ and $y'_i \geq y'_j$ for all $1 \leq i < j \leq n$. Then y majorizes x , denoted $x \prec y$, if

$$\sum_{i=1}^k x'_i \leq \sum_{i=1}^k y'_i \quad \text{for } j = 1, \dots, n.$$

Theorem 2.4: [30] Let H be an $n \times n$ Hermitian matrix with diagonal entries h_1, \dots, h_n and eigenvalues $\lambda_1, \dots, \lambda_n$. Then $h \prec \lambda$.

Proposition 2.5: [30] If $x, y \in \mathbb{R}^n$ and $x \prec y$, then $\prod_{i=1}^n x_i \geq \prod_{i=1}^n y_i$.

Corollary 2.6: Let A be an $n \times n$ Hermitian matrix, satisfying

$$\sum_{j=1, \dots, n, j \neq i} |A_{ij}| \leq \epsilon \text{ for all } i.$$

Then

$$\left| \sum_{1 \leq i \leq n} \log(1 + A_{ii}) - \sum_{1 \leq i \leq n} \log(1 + \lambda_i) \right| \leq n \log(1 + \epsilon).$$

Proof: First, by Theorem 2.4 and Proposition 2.5, $\prod_{i=1}^n A_{ii} \geq \prod_{i=1}^n \lambda_i$, and hence $\sum_{1 \leq i \leq n} \log(1 + A_{ii}) \geq \sum_{1 \leq i \leq n} \log(1 + \lambda_i)$. By the Gershgorin Disc Theorem [31] there exist ϵ_i such that $A_{i,i} = \lambda_i + \epsilon_i$ and $|\epsilon_i| \leq \epsilon$ for all i . By Proposition (2.5),

$$\begin{aligned}
&\log \prod_{i=1}^n (1 + A_{i,i}) - \log \prod_{i=1}^n (1 + \lambda_i) \\
&= \log \prod_{i=1}^n \left(\frac{1 + \lambda_i + \epsilon_i}{1 + \lambda_i} \right) \\
&\leq \log \prod_{i=1}^n \left(1 + \frac{\epsilon}{1 + \lambda_i} \right) \\
&\leq n \log(1 + \epsilon)
\end{aligned}$$

Lemma 2.7: Consider water-filling performed on two SNR sequences $\{N_1, \dots, N_L\}$ and $\{M_1, \dots, M_L\}$ with common total power constraint P . If $|N_l - M_l| \leq \epsilon$ for all l , then the water-filling allocations $\{P_1^N, \dots, P_L^N\}$ and $\{P_1^M, \dots, P_L^M\}$ satisfy $|P_l^N - P_l^M| \leq (L+1)\epsilon$ for all l .

Proof: Recall that for a real number x , $(x)^+ = \max(0, x)$. Define $\phi^N(x) = \sum_{l=1}^L (x - N_l)^+$ and $\phi^M(x) = \sum_{l=1}^L (x - M_l)^+$. If $|N_i - M_i| \leq \epsilon$ for all i , then $|\phi^N(x) - \phi^M(x)| \leq L\epsilon$ for all x . Both ϕ^N and ϕ^M are strictly increasing for $x > N_i$ and $x > M_i$ respectively. If, without

loss of generality, $\phi^N(x_0) = P$ and $\phi^M(x_0) < P$, then $\phi^N(x_0) - \phi^M(x_0) < L\epsilon$. Since the slope of ϕ^M is at least one, $\phi^M(x_0^M) = P$ for some $x_0^M < x_0^N + L\epsilon$. Let $P_l^N = (x_0^N - N_l)^+$ and $P_l^M = (x_0^M - M_l)^+$. Then $|P_l^N - P_l^M| \leq (L+1)\epsilon$. ■

Lemma 2.8: Consider a standard matrix channel given by $y = Ax + n$, where $A : \mathbb{C}^m \rightarrow \mathbb{C}^n$ and $n \sim \mathcal{NC}(0, \eta^2 I)$ and $\eta^2 > 0$. Let the eigenvalues of A^*A be denoted $\{\lambda_1, \dots, \lambda_n\}$ and let the power allocated according to water-filling for the SNR values $\{\frac{\eta^2}{\lambda_1}, \dots, \frac{\eta^2}{\lambda_n}\}$ be $\{P_1^\lambda, \dots, P_n^\lambda\}$. Let water-filling also be performed on the approximate SNR values $\{\frac{\eta^2}{\mu_1}, \dots, \frac{\eta^2}{\mu_n}\}$, where $\{\mu_1, \dots, \mu_n\}$ are approximations to the eigenvalues of A^*A , and denote this power allocation $\{P_1^\mu, \dots, P_n^\mu\}$. If $|\lambda_l - \mu_l| < \epsilon$ for all $l = 1, \dots, n$, then also for $l = 1, \dots, n$,

$$|\log(1 + \frac{P_l^\mu \mu_l}{\eta^2}) - \log(1 + \frac{P_l^\lambda \lambda_l}{\eta^2})| = \log(1 + (n+1)\mathcal{O}(\epsilon)).$$

Proof: The values N_l from Lemma 2.7 correspond to $N_l = \frac{\eta^2}{\lambda_l}$ and $M_l = \frac{\eta^2}{\mu_l} = \frac{\eta^2}{\lambda_l + \epsilon_l}$ where $|\epsilon_l| \leq \epsilon$. Then $|N_l - M_l| = \frac{\eta^2}{\lambda_l} |\frac{\epsilon_l}{\lambda_l + \epsilon_l}|$. Let $P_l^\mu = P_l^\lambda + \delta_l$. By Lemma 2.7, $|\delta_l| \leq (n+1)\frac{\eta^2}{\lambda_l} |\frac{\epsilon_l}{\lambda_l + \epsilon_l}|$. Now

$$\begin{aligned}
&|\log(1 + \frac{P_l^\mu \mu_l}{\eta^2}) - \log(1 + \frac{P_l^\lambda \lambda_l}{\eta^2})| \\
&= |\log(\frac{\eta^2 + P_l^\mu \mu_l}{\eta^2 + P_l^\lambda \lambda_l})| \\
&= |\log(1 + \frac{P_l^\lambda \epsilon_l + \delta_l(\lambda_l + \epsilon_l)}{\eta^2 + P_l^\lambda \lambda_l})| \\
&\leq |\log(1 + \frac{P_l^\lambda |\epsilon_l|}{\eta^2 + P_l^\lambda} + \frac{1}{\lambda_l} \frac{(n+1)\eta^2 \epsilon}{\eta^2 + P_l^\lambda})| \\
&\leq |\log(1 + (\frac{P_l^\lambda}{\eta^2 + P_l^\lambda} + \frac{1}{\lambda_l} \frac{(n+1)\eta^2}{\eta^2 + P_l^\lambda}) \epsilon)|.
\end{aligned}$$

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